Using ARMA Models in Stochastic Enterprise Valuation

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Abstract

This article presents a method for estimating the variance of the firm or enterprise value distribution by incorporating temporal dependencies in cash flows using ARMA models. The analysis highlights the importance of considering these dependencies, as neglecting them can lead to a significant increase in variance and subsequent erroneous decision-making. By utilizing ARMA models, decision-makers can obtain a more accurate assessment of the underlying risks and make informed investment decisions based on a comprehensive understanding of the firm's value distribution. The proposed method provides valuable insights for evaluating the uncertainty associated with future cash flows and enhances the accuracy of investment decision processes.

Keywords: *firm valuation, time series analysis, autocorrelation, investment decision-making, risk management in business.*

1. Introduction

This paper introduces an innovative approach to advancing stochastic enterprise valuation through the integration of Autoregressive Moving Average (ARMA) models. The analysis underscores the pivotal role of temporal dependencies in cash flows and illustrates the potential repercussions of disregarding them in decision-making processes. Employing ARMA models enables decision-makers to attain a more precise estimation of the variance in the firm's value distribution, offering valuable insights for enhanced risk assessment. Commencing with an overview of the foundational principles of stochastic enterprise valuation, the paper acknowledges the uncertainties linked to future cash flows. The Discounted Cash Flow (DCF) method is examined, treating cash flows as random variables to ascertain the distribution of the firm's value. The significance of accounting for autocorrelation in influencing factors is emphasized, demonstrated through a mathematical representation of the valuation process.

The paper delves into two scenarios: one with non-autocorrelated cash flows and another with autocorrelated cash flows. The latter involves the incorporation of ARMA models, specifically AR(1), MA(1), and ARMA(1,1) processes, to realistically address temporal dependencies[1]. Formulas are presented for calculating the expected value and variance of the enterprise value under these different models, highlighting the impact of autocorrelation on risk assessment.

Tables and figures illustrate the variance multipliers, accentuating the underestimation of variance in the presence of autocorrelation. Practical considerations for estimating parameters and the potential consequences of assuming independence or non-correlation are discussed. In conclusion, the paper advocates for a stochastic perspective in firm valuation, particularly in estimating the value of a company under uncertainty. The proposed method using ARMA models contributes to a more accurate risk assessment, preventing the underestimation of variance that may lead to flawed investment decisions. The findings underscore the importance of incorporating temporal dependencies for a robust evaluation of the uncertainty associated with future cash flows, ultimately facilitating informed decision-making in investment scenarios.

[1] For an in-depth exploration of specifying dependencies in stochastic firm valuation and the introduction of key concepts, refer to, for instance, Jöckel and Pflaumer (1981b), where a foundational understanding was extensively discussed.

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While subjectively specifying correct dependencies may be challenging, the paper notes that long time series data can aid in estimating dependencies through statistical methods.

Stochastic enterprise or firm evaluation estimates the value of a company by considering uncertainties and risks associated with future cash flows (see, e.g., [1] or [5]). It models future profits as random variables following a probability distribution. The firm's value is then calculated based on the expected present value of these cash flows, accounting for unpredictable market conditions and other factors. Decision-making under uncertainty has become crucial in firm valuation, as future results are significant but unknown[2]. Stochastic firm valuation incorporates statistical distributions to address uncertainties, enabling a more comprehensive analysis. The discounted cash flow (DCF) method, which sums discounted future cash flows, is commonly used. By treating cash flows as random variables, the distribution of the firm's value can be determined, aiding decision-makers in aligning with their risk attitudes. Neglecting autocorrelation in influencing factors carries the risk of making flawed decisions, emphasizing the importance of its consideration. The following formula provides a mathematical representation of the valuation process[3]:

$w_T = \sum_{t=1}^{T} v^t g_t$ with w_T = firm value g_t = free cash flow or free cash flow to firm (FCF, FCFF) $v = \frac{1}{1+i}$ discounting factor i = interest rate $t = 1, 2, ..., T$ period

2. Distribution of Enterprise Value with Non-Autocorrelated Cash Flows

The objective is to value a company. The valuator is required to make estimates of the future expected cash flows which can be subjective or objective in nature but share the characteristic of being uncertain or subject to errors. To account for this, the relevant returns g at time t: 1, 2, ..., are represented as: $g_t = \mu_t + u_t$ with $E(u_t = 0)$ $E(u_t u_s) = \begin{cases} \sigma^2 & t = s \\ 0 & otherwise \end{cases}$ where μ_t, the mean function, reflects the deterministic part, and u_t represents the stochastic part of the earnings. Assuming infinite lifespan, the enterprise value is given by: $w = \lim_{T \to \infty} w_T$ if existing. If the stochastic part of the earnings stream is non-trivial, i.e., $Var(u_t) \neq 0$, the question arises: what is the probability that the random variable w falls within a given interval $[a,b]$. To calculate the probability $P(a \le w \le b)$ distributional assumptions about the disturbance variable u_t are required. The simplest case is that of normal distribution, $u_t = N(0, \sigma^2)$ for t=1,2,… This implies that the enterprise value w follows a normal distribution with the expected value: $Ew = \sum_{t=1}^{\infty} v^t g_t$ (if the series converges) and the variance: $Var\, w = \frac{v^2}{1-v^2}\sigma^2$. The probability that w lies between a and b is given by:

$$P(a \le w \le b) = \phi\left(\frac{b - \sum_{t=1}^{\infty} v^t \mu_t}{v\sigma} \sqrt{1-v^2} \right) - \phi\left(\frac{a - \sum_{t=1}^{\infty} v^t \mu_t}{v\sigma} \sqrt{1-v^2} \right).$$

where ϕ represents the distribution function of the standard normal distribution.

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[2] See also risk analysis in capital budgeting or investment appraisal (Pflaumer, 2017, Savvides, 2008, Vose, 1997, Jöckel and Pflaumer, 1980 and 1981a.

[3] For the calculation of cash flows (FCF or FCFF) and the determination of the appropriate discount rate, see Damadoran, 2006, Hitchner, 2017 or Fazzini, 2018.

For small interest rates i, the above equation is also approximately valid for arbitrarily independent identically distributed u_t (see [7], [8]). The assumption of normal distribution for the residuals should be critically assessed. However, this assumption is not as restrictive as one might initially suspect. The normal distribution is a suitable mathematical model for many phenomena encountered in reality, assuming that numerous influencing components act independently and additively. Since the residuals are indeed influenced by numerous factors, both internal and external to the company, and in many cases are more or less independent, it is not excluded that the normal distribution is an appropriate distributional law in the present case. Even if a different distributional law is assumed for the disturbance variables, it does not affect the expected value and variance of the distribution of enterprise value. These parameters are independent of the residual distributions. Regarding the mean function μ_t , representing the systematic part of the surplus, it is assumed that it can be adequately approximated by a known class of functions of time $f_1(t)$, $f_2(t)$, ..., $f_n(t)$: $\mu_t = \mu + \sum_{t=1}^{h} \beta_j f_j(t)$ for t= 1,2,….

Table 1 provides examples of different mean functions and the corresponding expected values for enterprise value. A generalization can be achieved when the variable explaining the earnings g is represented by the mean function μ_t . The earnings stream at time t is obtained as: $g_t = \mu_t + u_t$ for t=1,2,… with $\mu_t = \beta_0 + \beta_{1,t} x_{1,t} + ... \beta_{k,t} x_{k,t}$ if a linear regression model is assumed. The regressors x are influencing factors that determine the relevant earnings g, and they can be of both operational and macroeconomic nature. Interdependencies between the variables can be accounted for by constructing simultaneous econometric models. Assuming that the parameters β_i are stable, the enterprise value could be determined if the values of all regressors or all predetermined variables were known. However, the key challenge lies in the determination of the predetermined variables. This problem has been shifted and, in general, made more difficult. Instead of predicting one variable, namely the earnings g, one now needs to forecast *k* variables, which is certainly not easier than predicting earnings. For practical applications, it is typically assumed that the mean function is either a simple function of time or even constant.

Table 1: Examples of Mean Functions of Time

Mean function μ_t **for** ***t*****=1,2,…**		**Expectation *Ew***
1. Constant function	$\mu_t = \mu$	$Ew = \frac{v\mu}{1-v} = \frac{\mu}{i}$
2. Linear Trend	$\mu_t = \mu + \beta_t t$	$Ew = \frac{\mu}{i} + \frac{v\beta_1}{\left(1-v^2\right)}$
3. Geometric Trend	$\mu_t = \mu\left(1+g\right)^t \quad g < i = growth\,rate$	$Ew = \frac{\left(1+g\right)\mu}{\left(i-g\right)}$
4. Periodic Function	$\mu_t = \mu_{t+N} \quad N = period$	$Ew = \frac{1}{1-v^N}\sum_{i=1}^{N} v^t \mu_t$
5. Saturation Curve	$\mu_t = \exp\left(a + \frac{b}{i}\right) \quad a<0,\, b<0$	$Ew = e^a \sum_{i=1}^{\infty} v^t e^{\frac{b}{t}}$

3. Distribution of Enterprise Value with Autocorrelated Cash Flows

The previous explanations assume the stochastic independence of relevant cash flows. From a practical perspective, assuming independent identical distributions - even more so than assuming normal distribution - is too restrictive. In reality, economic variables are dependent random variables. This means that, under the assumption of normal distribution, the covariance or correlation between cash flows in different periods is nonzero.

A first step towards incorporating this temporal dependence into enterprise valuation and achieving a more practical solution is to assume a stationary stochastic process for the disturbance variable u_t in the form of an ARMA (p, q) process: $u_t = \phi_1 u_{t-1} + \phi_2 u_{t-2} + ...\phi_p \mu_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + ... + \theta_{t-q} \varepsilon_{t-q}$

The first part of the process is referred to as the autoregressive (AR) component, and the second part as the Moving Average (MA) component. Here, ϕ_i and θ_i denote the parameters of the process, and ε_t represents the disturbances assumed to be independent, normally distributed random variables[4].

The model assumptions made for u_t allow us to consider a large class of models. To avoid an excessively large number of parameters that need to be estimated, which would render the procedure impractical, three simple examples of different processes u_t are examined. These include the autoregressive process of order one, AR(1), the Moving Average process of order q, MA(q), and the mixed process ARMA(1,1). In the presence of a Moving Average process of order one, the disturbance process depends on the disturbance variable ε_t and the disturbance variable of the previous period ϵ_{t-1}. This means that the

[4] Details of ARMA and ARIMA modeling can be found, e.g., in Chatfield, 2004; Jöckel and Pflaumer, 2022 used these models to analyze and forecast mortality and excess mortality during the Corona pandemic; in Jöckel and Pflaumer, 1981c, a notable early application of the Box-Jenkins method in German scientific literature, monthly gold prices are forecast.

disturbance has an impact exactly one period later. Specifically, for example, a parity change[5], assumed to have a stochastic effect, influences the dependent variable in the period it occurs and in the subsequent period. A disadvantage of using a higher-order Moving Average process to capture temporal dependencies is that it may require specifying or estimating a large number of parameters. On the other hand, disturbance processes with the aftereffects of disturbance variables in one period affecting the disturbance variables in all subsequent periods can be represented by autoregressive processes. The autoregressive process of order one $u_t = \phi_1 u_{t-1} + \varepsilon_t \quad \left[|\phi_1| < 1 \right]$ can be represented as an infinite Moving Average process, where the influence of a disturbance diminishes over time $u_t = \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2} + \phi_{1\,q}^3 \varepsilon_{t-3} + ...$ It is important to note that only a single parameter, ϕ_1, needs to be estimated, and all intertemporal covariances are nonzero. To model the disturbance process as realistically as possible, one cannot assume that the aftereffects of a disturbance decrease exponentially immediately, as the autoregressive process of order one assumes. A solution is to combine the two types of disturbance processes. This has the advantage of being able to weight delays of some periods using a Moving Average process and, at the same time, taking into account the less significant dependencies of the remaining periods through an autoregressive process with few parameters to be specified.

The ARMA (1,1) process $u_t = \phi_1 u_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1} \quad \left[|\phi_1| < 1 \ and \ |\theta_1| < 1 \right]$ (1)

emphasizes the significance of the aftereffects of one period without neglecting the remaining dependencies. Now, each process of the form (1) can be represented as: $u_t = \sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-k}$ with certain α_k (k: 0, 1, 2, ...) and stochastically independent $N(0, \sigma_\varepsilon^2)$-distributed ε_t (t: 0, $(t = 0, \pm 1, \pm 2...)$is called the Moving Average representation of (1). Using (2), the expected value and covariance function of u_t can be calculated. It holds that:

$$E u_t = 0, \ \gamma_0 = Var\, u_t = \sigma^2 \sum_{k=0}^{\infty} \alpha_k^2, \ \gamma_l = Cov(u_t, u_{t+l}) = \sigma_\varepsilon^2 \left(\sum_{k=0}^{\infty} \alpha_k \alpha_{k+l} \right) \quad l = 1, 2, ...$$

With the help of these formulas, the distribution of the enterprise value for a general process of the form (2) can be derived by some calculus. The result is a normal distribution with the parameters $E\, w = \sum_{t=1}^{\infty} v^t \mu_t$ and $Var\, w = \frac{v^2}{1 - v^2} \left(\gamma_0 + 2 \sum_{k=1}^{\infty} v^k \gamma_k \right)$. The variance of the enterprise value depends not only on the variance of the disturbance process but also on its covariances. Neglecting the temporal dependence in performance indicators can lead to a significant underestimation of the variance of the enterprise value, particularly in the case of positive autocorrelation - in economic variables, positive autocorrelation is more likely than negative autocorrelation. The extent of underestimation depends, as clearly shown in Table 2, on the specific residual process and its parameters.

The numbers in the cells of Table 3, which we want to refer to as risk factors or variance multipliers, reflect the ratio of variances of the distributions of firm values under alternative assumptions about the parameters of the disturbance processes and interest rates. They are derived from Table 2. The reference value is the variance of the firm value under the assumption of temporal independence. The temporal dependencies are represented in the variance multipliers in the first row of Table 3 by an AR(1) process, in the first column by an MA(1) process, and in the remaining cells by an ARMA(1,1) process. If the interest rate

[5] Parity change in time series analysis refers to a shift or alteration in the relationship between variables over time, and it is crucial to detect and account for it to ensure accurate modeling and forecasting.

is 20% and the underlying disturbance process is an ARMA(1,1) model with $\phi = 0.9$ and $\theta = 0.5$, the resulting variance multiplier is 7.29. Failure to consider this dependence in the profits would have led to an underestimation of the variance of the firm value by a factor of 7.29. The variance multipliers decrease as interest rates increase. Similar to how the expected value decreases with increasing interest rates, the same is true for the variance.

Table 2: Expected Values and Variances of the Enterprise value for different Models

Disturbance Process	**Expectation**	**Variance**
White Noise $u_t = \varepsilon_t$	$E\,w = \sum_{t=1}^{\infty} v^t \mu_t$	$\frac{v^2}{1-v^2}\gamma_0$
MA(1) Process $u_t = \varepsilon_t + \theta_1 \varepsilon_{t-1}$	$E\,w = \sum_{t=1}^{\infty} v^t \mu_t$	$\frac{v^2}{1-v^2}\gamma_0 \left(1 + 2\frac{\theta_1}{1+\theta_1^2}\right)$
AR(1) Process $u_t = \phi_1 u_{t-1} + \varepsilon_t$	$E\,w = \sum_{t=1}^{\infty} v^t \mu_t$	$\frac{v^2}{1-v^2}\gamma_0 \left(\frac{1+v\phi_1}{1-v\phi_1}\right)$
ARMA(1,1) Process $u_t = \phi_1 u_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$	$E\,w = \sum_{t=1}^{\infty} v^t \mu_t$	$\frac{v^2}{1-v^2}\gamma_0 \left(1 + 2\alpha v + 2\alpha\phi_1 v^2 \frac{1}{1-\phi_1 v}\right)$ with $\alpha = \frac{(1+\phi_1\theta_1)(\phi_1+\theta_1)}{1+\theta_1^2+2\phi_1\theta_1}$

Since, for practical reasons (often due to sufficiently long time series required for estimating ARIMA models), it is almost impossible to adequately determine ϕ and θ, we often assume independence or non-correlation. However, this carries the risk of underestimating the investment risk. As evident from the tables and the plots in Figs. 1 and 2, the risk primarily depends on ϕ and to a lesser extent on θ, making it reasonable to focus solely on the subjective estimation of ϕ. In most cases, it is impossible to subjectively specify correct dependencies. Practical solutions involve calculating the firm value assuming independence and weak autocorrelation (phi = 0.3), medium autocorrelation (phi = 0.5), or strong autocorrelation (phi = 0.8) of the influencing factors to obtain a range of resulting variances.

Table 3: Variance multipliers for an interest rate of 20%

$\theta \backslash \phi$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
0	1	1.18	1.4	1.67	2	2.43	3	3.8	5	7	11
0.1	1.17	1.36	1.58	1.86	2.19	2.62	3.19	3.98	5.15	7.11	11
0.2	1.32	1.52	1.74	2.02	2.35	2.77	3.33	4.11	5.26	7.18	11
0.3	1.46	1.65	1.88	2.14	2.47	2.89	3.44	4.21	5.34	7.23	11
0.4	1.57	1.76	1.98	2.24	2.57	2.98	3.52	4.27	5.4	7.27	11
0.5	1.67	1.85	2.06	2.32	2.64	3.04	3.58	4.32	5.44	7.29	11
0.6	1.74	1.91	2.12	2.37	2.68	3.08	3.62	4.36	5.47	7.31	11
0.7	1.78	1.95	2.16	2.41	2.72	3.11	3.64	4.38	5.48	7.32	11
0.8	1.81	1.98	2.18	2.43	2.74	3.13	3.66	4.39	5.49	7.33	11
0.9	1.83	2	2.2	2.44	2.75	3.14	3.66	4.4	5.5	7.33	11
1	1.83	2	2.2	2.44	2.75	3.14	3.66	4.4	5.5	7.33	11

This multiplier indicates that the variance is underestimated by a factor of 7.29, e.g., if temporal dependencies are not taken into account. As a decision-maker, one should calculate the loss probabilities for each variance assumption. If the loss probability remains low even with high autocorrelation (phi = 0.8), then the enterprise value is justifiable from the buyer's perspective. Conversely, from the seller's perspective, one should calculate the probabilities of the enterprise value being greater than, for example, the 0.95 quantile. If the probability is high, it indicates that the demanded selling price of the enterprise may have been set too low.

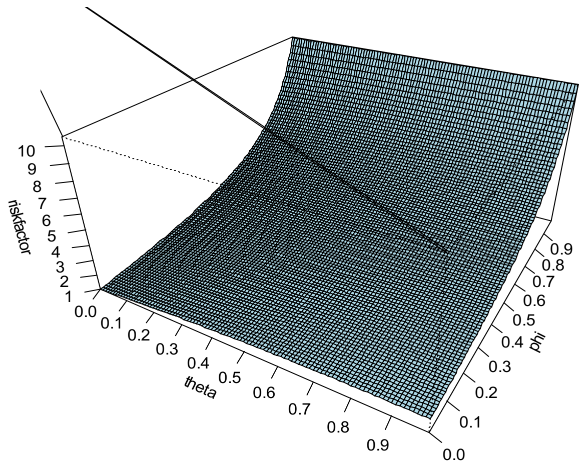

Figure 1: Surface Plot of Risk Factors (Variance Multipliers) with Interest Rate 20% (p=0.2)

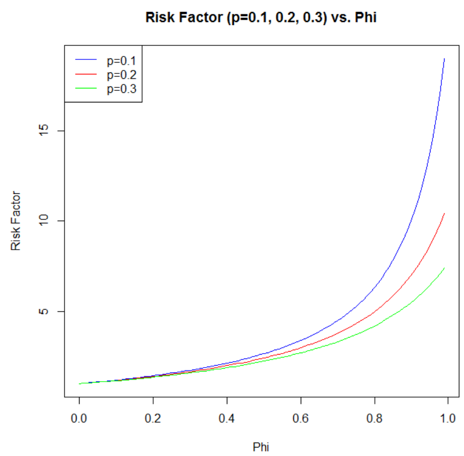

Figure 2: Risk Factors (Variance Multipliers) as a Function of phi (theta=0) and Interest Rates p (0.1, 0.2, 0.3)

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